Lecture 2

Basic Induction

Prep

* PowerPoint for triangles
* Beaver Flu on PS2
* 

on PS2 or in recitation

* Write bigger; fewer lines on board

Take

* Laptop with PowerPoint for triangles
* Color chalk

**Reminder**

**PS #2 on website, due on Monday 7:30 PM \*\***

**Read 2.5 – 2.7 and 3.0 – 3.4 (sec 3.5 is optional)**

**10-pt bonus on PS 2 for seeing TA during office hours.**

Want to encourage you to take advantage of teaching assistants and office hours. This week only. Can see any TA but best to see your TA to get to know them.

Also good opportunity to get hint on Beaver Flu problem \*\*

If there is anyone that has not hooked up with a recitation instructor, please see staff \*\* after class.

Last week, we talked about the key components of a proof: propositions, axioms, and logical deductions. As you probably discussed in recitation, we’re not going to worry very much about what axioms and logical deductions you use in your proofs – just make sure that they are reasonable. For the most part, you can assume any basic facts you knew about math coming into the class. Of course, if we ask you to prove some proposition P for homework or a test, you can’t say “I already knew P so it is an axiom” – generally we want you to prove P from some more elementary facts. Also, you don’t have to worry about saying you used modus ponens or some other rule when making a step in a proof – just make sure that your deductions are logical and easy to follow. Where you can get into trouble is if you make wild leaps of faith or say “the proof is left to the reader”. That is not so good.

Questions?

The proofs that we covered last week in recitation and homework were all examples of direct proofs. Direct proofs are often the simplest – you start with some axioms & previously proved theorems and build on them with deductions until you get the desired result. Sometimes, however, it is easier to use an indirect proof, which is also known as a proof by contradiction.

In a proof by contradiction, you assume the opposite of what you are trying to prove and try to derive a falsehood or contradiction. If you start with an assumption and then get to something that is untrue, then it must be that your assumption was untrue. Let’s write that down.

**Proof by contradiction**

**To prove P is true, we assume P is F (**which is the same as assuming **¬P is T) and then** we use that assumption to **derive a falsehood** or contradiction**.**

If you derive a contradiction, then it must be that P is not False – which means that P is true. This works because

**If “¬ P ⇒ F” is True**,then we know from the definition of implies that

**“¬ P ⇒ F”**

**F** which means that

P is True

Questions?

Ok let’s do an example. Let’s prove:

**Thm: is an irrational number**

**P**

**Q:** Doeseveryone know what irrational numbers are? They are numbers that can’t be expressed as a ratio of integers.

1. Everyone knows  is irrational, right. How many know a proof? Raise your hands… OK, let’s prove it.

**Proof:** If you tried to do a direct proof, it would be pretty hard – how do you show you can’t represent  as the ratio of integers without trying all of them? -- there are a lot of integers to rule out! But the proof is pretty easy if we do a proof by contradiction.

When doing a proof by contradiction, you always start by letting the reader know that is what you are doing. So you write:

**Proof: (by Cont.)**

The next thing you do is to assume that what you are trying to prove is False. So you write that down:

**Assume for purpose of contradiction that is rational**

**¬P**

**Label P and ¬P and explain setup**

Let’s see where this leads us.

**⇒ ∃ a, b ∈N + s.t. = a/b where a & b have no common divisors.** In other words, a/b is in lowest terms -- if there were any common divisors, we could just cancel them out until we get a and b without common divisors. This is a well-known fact about rational numbers.

**⇒ 2 = a2/ b2**

**⇒ 2b2 =a2**

**Q:** What does this imply about a?

**A:** Even

**⇒ a is even** which means that 2 divides a; written **(2|a)**

**Q:** What does this imply about a2 ? (We know more than just that it is even.)

**A:** Mult of 4

**⇒ 4 |a2**

**⇒ 4 | 2b2**

**⇒ 2 | b2**

**Q:** What does this imply about b?

**A:** b even

**⇒ b is even (2|b)**

**Q:** Do we have a contradiction?

1. Yes!

**Q:** Why is this a contradiction?

1. a/b are both even and so they share a common factor which is known to be False.

**⇒ a/b share a common divisor** #

**Contradiction** symbol

So we conclude that the original assumption must be wrong, which means that

**⇒ is irrational**

Box, check, QED – all mean that the proof is over now.

Questions?

Not too hard a proof. But nice.

Actually, there is an interesting history behind this proof. As far as we know it was first discovered by the Pythagoreans. The Pythagoreans were a religious society started by Pythagoras in Ancient Greece.

Sounds pretty weird today, but in Ancient Greece, math was considered as sort of a religion that was ruled by Gods. Now every once in a while, you’ll see someone at MIT who thinks math is a religion, but in Ancient Greece, it really was! The 2 most important math Gods were Apeiron & Peros.

Apeironwas the God of infinity and represented all that was bad. Probably that’s because infinity is a really tricky concept for humans to grasp. Even mathematicians struggle with it.

On the other hand, finiteness is nice, and in Ancient Greece, Peros was the God of finite world and represented all that was good.

One of the main beliefs or axioms of the Pythagoreans was that there were no irrational numbers. They simply did not exist. The reason was that irrational numbers were bad because they were “infinite” in the sense that they could not be expressed as ratio of integers or even as a finite repeating string. E.g., 1/7 is ok since finite repeating sequence.

**1/7 = .142857**

In fact, finiteness was so important that they had an axiom that said that every line had a numeric length. Now, Pythagoreans were good geometers & they knew that if you took right triangle with unit sides, then the length of hypotenuse was **.** (Pythagorean Thm).

**1 **

**1**

Given the axioms that every line has a numeric length & that all numbers are rational, you can easily prove that **** is rational.

Things were going along fine in Ancient Greece until one day when they discovered the proof that  was not rational! That caused a big problem because it meant that their axioms were not consistent, and that meant that all their proofs were bogus. Maybe even worse, it meant that the devil () was in their midst.

Sort of like discovering that there were only 9 commandments and then that the devil had added the 10th commandment! And, worse, you aren’t sure which one the devil put there. Sacrilege!

So what were they to do? Well, they did the natural thing: they covered it up and denied the existence of the result! They kept saying that  was rational!

According to legend, however, there was a deep throat among the Pythagoreans who let the word out … and so they killed him! This is the legend anyway. Hard to imagine getting killed over the irrationality of, but who knows ….

You’ll see a lot of proofs by contradiction in homework and during the term. The next proof technique I want to cover is one of my favorites & that is a false proof. We’ll also see a lot of these during the term.

I am going to prove to you that 90 > 92. See if you can spot the flaw.

**Go to Power Point**

Only thing worse than proof by picture is proof by Power Point! **☺**

**Do Proof**

Yikes! What went wrong?

Now, I made up an axiom about conservation of area – maybe it’s not consistent with arithmetic & geometry? Hand up if you think that is a bad axiom. If so, then I could keep on going and turn 90 square inches of gold into an infinite amount of gold – just keep repeating! **-Explain-**

Or maybe axioms of math that we all know and love are really inconsistent and it has been covered up all these years! (just like Pythagoreans). Maybe math is just bogus.

Raise your hand if you think math is bogus. ☺

I was afraid of that. **☺**

One of the nice things about proofs is that when there is a bug, you can usually spot it by going over each step very carefully – of course, all the steps must be specified and stated clearly – often it’s a step that is glossed over that contains the bug! And glossing things over is super easy to do when you use pictures in a proof. That’s why it is always better to write out all the steps—much easier to find the bug that way.

Q. What is BUG??

Well, the picture got me into trouble at the very start when I labelled it so that the side labeled 9 was longer then the side labeled 10. Easy mistake to make. Now the side labeled 2+ really is longer than the side labeled 2, but the math gets fouled up because of the mistake with 9 and 10.

Show on next slide

If you put in the correct lengths (swapping 9 and 10) and do the math, you will find that the area is in fact at least 88, which makes sense since 90 is at least 88.

This is how Kempe’s “proof” of 4 – color thm lasted 10 years before bug found. It happens all the time in the literature. You start a proof by saying it looks like “this” and you are dead before you can say “oops”!

Questions?

For rest of today and this week, we are going to talk about proofs by induction. Induction is by far the most powerful and commonly used proof technique in computer science. If there is one thing you should know by the time you are done with this class – it is how to do a proof by induction.

Actually, I should rephrase that – if there is one thing you will know by the time you are done with this class, it is how to do a proof by induction. And just to make sure that you become intimately familiar with induction, we will be spending next 5 – 6 weeks using induction to analyze and solve all sorts of problems.

We’ll probably do at least a dozen inductive arguments in class and many more for homework. And, of course, there will be at least one induction problem on every test. You will probably be dreaming about induction before we’re through with you.

The good news is that induction is very simple and very easy to use. Technically speaking, induction is just an axiom which can be stated as follows.

**Induction Axiom**

**Let P(n) be a predicate. If P(0) is true and**

**SAVE**

**“∀n∈N P(n) ⇒ P (n+1)” is true, then “∀n P(n)” is true.**

Another way of saying this without the for-alls is

**(I.e., if P(0) is true, and P(0) ⇒ P (1), P (1) ⇒ P (2), … , are all true, then P(0), P(1), P(2), … , are all true.**

forever

From the latter formulation, it should be clear why induction is a reasonable axiom. For example, if P (0) is true and P(0)⇒P(1) is true, then P(1) must be true by modus ponens. And then if

P(1)⇒P(2) is true, then by modus ponens, P(2) must be true, and so on forever. We keep building up implications to show P(n) is true for all n.

Another way to see this is to think of the sequence of propositions P(n) as a sequence of dominos where we knock down the nth domino if P(n) is true.

**P(n+1)**

**P(n)**

**........**

**.........**

**P(3)**

**P(2)**

**P(1)**

**P(0)**

We are given that P (0) is true so we knock it down. Each falling domino knocks over the next since we are given P (n) ⇒ P (n+1) for all n. Hence all dominos are knocked down and ∀n∈P (n) is true.

Questions?

**Q:** Raise your hand if you don’t have much experience with induction? We are going to change that… ☺

Let’s do a simple example of a proof using induction.

**Thm: ∀ n ≥ 0 1+2+3+ …+ n = **

This is actually a pretty useful fact to remember and we’ll use it alot this term..

Before we prove this fact, let’s make sure we understand what the notation on LHS means. It’s the source of a lot of errors and can get folks confused.

“…..” indicate omitted terms and it is assumed that enough of the pattern is supplied for us to fill it in. very handy shortcut and its used all the time. but this kind of imprecision is always dangerous since it may not be absolutely clear what the pattern is.

Here, we mean sum of first n positive integers.

Because it is imprecise, mathematicians have created a special notation for sums like this:



So there are four different ways to write the sum of the integers from 1 to n and we will use them all during the term.

Couple of special cases for the sum which we should be careful about.

⇐ No 2! Be careful

⇐ Just one 1, even tho it seems like 2 of them!

⇐No 1 or 2

**If n = 1 Q/A 1 + 2 + … + n = 1 (1 term)**

**If n ≤ 0 Q/A 1 + 2 + … + n = 0 (0 terms)**

Often cause of mistakes, as we will see later today.

Questions?

OK, back to our theorem. It’s easy enough to check the result for any particular value of n, say **n=4: 1+2+3+4=10 =** , but to know it is true for all n, we will need a proof. There are several ways to prove it – but induction is easiest.

Whenever you are doing a proof by induction, you start by writing that down, just like with a proof by contradiction. This tells the reader what to expect in terms of what’s coming and the structure of the proof so they won’t get confused.

**Proof (by induction)**

The next thing to do is to figure out the predicate P(n) in induction axiom. Show P(n) in axiom

Often P(n) is what you want to prove. And that will turn out to work well in this case. So we

**Let P(n) be proposition that**  **=** .

**(Inductive Hypothesis)**

Must state what P(n) is - Must write it down.

Next we need to check that P(0) is true. This is called the Base Case or Basis step.

**Base Case: P(0) is true since**

**Q/A:** both sides 0

 **= 0 =** . No terms in this sum!

refer to axiom for P(0) and P(n) ⇒ P(n+1)

Next, we prove P(n) **⇒** P (n+1), it is called the inductive step.

**Inductive Step: For n ≥ 0, show P(n) ⇒P(n+1)**

**Q:** How do we show P(n) ⇒P(n+1)?

**A:** This is an implication, and from last time we know that we can prove it by showing that P(n+1) is true whenever P(n) is true. So for any n>0, we start by assuming that P(n) is true & then go through a series of logical deductions in order to derive that P(n+1) is true. (Remember that we can assume P(n) is true since it does not matter what happens when P(n) is false since then P(n) ⇒ P(n+1) is true by definition of ⇒.)

Questions?

So we write:

**Assume P(n) is true for purposes of Induction.**

Always good to say why you are assuming something is true – for example, in this case, it is not because we think it is an axiom or that we are assuming it for purposes of contradiction.

**Always remember to write this down!**

So in this case we

**(i.e., assume 1+2+ ….. + n =  )**

and we need to show that this implies that

**Need to show:**

**1 + 2 + ……. + (n+1) = **

Now this seems sort of weird. It looks like we just assumed what we are trying to prove which would be bogus, and this confuses folks who are not familiar with induction. But what we actually assumed was the it was true for a certain value of n, and now we want to use that to prove it is true for the next value of n (namely, n+1). So we are not assuming what we are trying to prove. We are trying to prove that if it is true for n, then it is also true for n+1.

Questions?

OK, so, let’s check if this works.

**P(n)**

**1+2+ ... +n+1 = (1+2+ ... +n) + (n+1)**

**=  +n+1** since we assumed P(n) ⇒P(n+1)

**= **

**=  =**

**= **

**⇒ P(n+1) is true**

Ok, we have shown P(n) ⇒ P(n+1). Since we did not make any restriction on n (other than ≥ 0) it works for all n ≥ 0 and we have verified that **so** “**∀n≥0 P(n) ⇒** **P(n+1)” is true** as desired.

We now invoke the induction axiom to conclude that

**So “∀n≥0 P(n)” is True by induction axiom** and this completes the proof.

Questions?

**Q:** Induction helped us prove the theorem, but did it help us understand why it is true? Do you have any feel for why the Theorem is true after seeing the proof?

**A:** Not really, maybe it gave us some insight that both sides grow at same rate as n increases. But not really a good understanding of why true. Sometimes Induction helps us understand why something is true and sometimes not. Later, we’ll see an example where it helps us understand why something is true.

**Q:** Did induction help us figure out the value of the sum?

**A:** No. In this case, we needed to be given answer before we could use induction. But sometimes induction can be used to help derive answer or solution as we’ll see in the next example. In fact, in the next example, we’ll see how induction can be used to solve a real problem and get a proof at same time.

This problem arose during the construction of the Stata Center. Now, the Stata center was originally supposed to cost about $100m. But MIT hired Frank Geary to be the architect and costs quickly spiraled out of control. Geary is very famous but also famously expensive -- it costs a ton of money to build all those bent walls at weird angles!! **☺**

Anyway, the construction costs quickly got way over $300M. This part is really true – but now I’m going to bend the truth just a little for the example. Anyway, fundraising became a huge priority and some radical ideas were proposed. One plan was to build a large 2nx 2n courtyard and put a statue of a wealthy potential donor in the center.

**Courtyard  2n**

**2n**

CS Courtyard – so must be power of 2, of course.

Now we needed to reserve one of the four center squares for the statue of a wealthy guy whose name I’m not supposed to reveal, so let’s just call him Bill.

**Bill**

**Ex n=2**

Now this would have been an easy task except that nothing was easy with Frank Geary. He insisted on using special L-shaped tiles instead of square tiles.

**2**

**L-shaped Tiles**

2

The problem was then to explain to the tile guy how to lay the tiles so that they make a perfect fit, while leaving room for Bill in center.

**Show an Ex.**

Now it works in this case with n=2, but how do we know it is possible to do it for an arbitrary2n x 2n courtyard? And how do we actually do the tiling? There is a big difference between knowing it is possible to do and knowing how to actually do it.

Is everyone clear on the goal?

Ok, let’s try using induction.

**Thm: ∀n, ∃ way to tile a 2n x 2n region with a center squaremissing** (for Bill)

**Proof: (by induction)** Tell the reader what proof method

**I.H. P(n)** Next step is to identify the IH

**Base case: P(0): 20 X 20** Never forget the base case!

**Bill**

**Inductive Step: For n ≥ 0, assume P(n) to verify I.H.**

**So we need to show P(n+1) is true.**

**Consider a 2n+1 x 2n+1 Courtyard**

**Explain**

**There are 2n+1**

**rows and columns**

**2n+1**

**2n+1**

**Q**: What should we do to be able to apply I.H. P(n)? Need 2n x 2n courtyards.

**A:** Divide in quarters. **Show in Blue**

But wait, what is the trouble with tiling 4 regions separately?

**Q:** What about quadrants without Bill?

**A:** IH does not apply! **Q/A** – it is even possible to do?

**A:** Not possible to do **Q/A** – Give Proof

Ex n=2, size 4 x 4 = 16 and 3 does not divide 16!

So this is not working ….

**Q:** Any ideas for overcoming this problem?

Getting right # tiles in quadrant

Hint: Add a L-shaped tile right around Bill.

Bill

Ok, now we can proceed with all four corners separately and try to use induction.

**Q:** Can we tile regions inductively?

**A:** No – Bill is not in center of the 2n + 2n courtyards, so IH does not hold.

In cases like this, it often helps to change the IH.

**Q:** How should we change I.H.?

**A:** Tiling possible if Bill is in corner? This works sort of.

**Q(n): ∃ tiling of 2n x 2n region with corner square missing**

**Q(n) ⇒ Q(n+1)**

**Bill**

**Explain Proof**

Questions?

**Q:** What is the only problem?

**A:** Bill does not want to be in the corner. He wants to be in the center so we solved a different problem – albeit related.

**Q:** How do we solve original problem with Bill in center?

**A:** Make Induction Hypothesis stronger. I.E., try to prove a stronger thm.

Sometimes it is easier to prove a harder result.

**Solution: “strengthen” P(n)**

**Ex: P(n): ∀ locations of Bill, ∃ a tiling of a 2n x 2n region around Bill.**

So Bill could be in center, corner, or anywhere he wants. Much harder result – but proof is much easier.

**Proof: Base Case: P(0) Bill** Only one option for Bill

**Induction Step:** **For n ≥ 0, assume P(n) to prove P(n+1)**

**Look at P(n+1)**

**2n+1**

**2n**

Bill

**2n**

**2n**

**2n**

**2n+1**

**Q/A** Where put 1st tile? At center

**Place 1 tile at center in 2n x 2n regions w/o Bill.**

**Tile each 2n x 2n region by induction.**

Questions?

Two very nice facts about this proof.

1st – the proof actually provides an algorithm for doing the tiling. Not only do we know a solution exists – but we can find it by applying proof. Just write recursive program based on inductive proof!

2nd - We got a stronger result – now Bill can be anywhere!

**Q:** That seems strange. Any idea why was it easier to prove a stronger result?

**A:** Proving P(n) ⇒ P(n+1) is sometimes easier if P is stronger.

There is more to prove in P(n+1) but there is more to work with when P(n) is assumed to be true.

Bizarre Concept – You’ve heard the expression “if at first you don’t succeed, then try try again”. Well with induction, “if you don’t succeed at first, then try something even harder!!”

We’ll see lots more examples like this. In fact, the whole name of the game with induction is to get the right Induction Hypothesis. Sometimes easy – sometimes not. Bit of an art to getting right I.H.

Questions?

Ok, let’s test your knowledge of induction by trying to prove something that is clearly false – see if you can spot the problem.

**Theorem (not!): All horses are the same color.**

**“Proof” (by induction)**

Next we need to identify the induction hypothesis . Usually, we try using the proposition we are trying to prove. In this case, that would be that all horses are the same color.

Q. Can we do that here?

A. No

Q. Why not? Why can’t we use “all horses are the same color” as the induction hypothesis?

A. There is no variable—we need a parameter like n to induct on.

But we can fix this as follows:

**P(n): In any set of n ≥ 1 horses, all the horses in the set have the same color.**

This will be good enough since if I prove it is true for all n, then it will be true for the set of all horses, which will mean that all horses have the same color.

Questions?

OK, next we do the base case. (This time we start with n=1 instead of n=0 but no real difference since the empty set of horses is not very interesting.)

**Base Case: P(1): true since only 1 horse.**

**Inductive Step: Assume P(n) to verify I.H.**

**Consider any set of n + 1 horses H1, H2, …… Hn+1**

Q. What do we know about the first n of these horses, H1 through Hn?

**A. H1, …. Hn are the same color.**

**Q** why?

**A**. because of Induction assumption P(n)

Q. and what about the last n horses H2 through Hn+1?

**A. H2, …. Hn+1 are the same color.**

**Q.** why?

**A.** For the same reason—they are just another set of n horses and the induction assumption says that any set of n horses have the same color.

**⇒ Since color (H1)= color (H2, …, Hn) = color (Hn+1),**

**⇒All n + 1 horses are the same color**

**⇒ P(n+1)**

Questions about proof?

A few years ago, we assigned this problem as homework and asked students to figure out what went wrong.

The responses were a little disappointing … ½ the class responded that “this example just goes to show that induction doesn’t always work”.

1/3 said “I always knew that you can’t trust mathematics this example just proves it”. That really hurt!

Most of the rest were variations on this theme. Not exactly what we were looking for in HW so now we do it in class. **☺**  Only a few figured out what the flaw was. It’s very subtle.

Ideas?

Proved P(1)

Proved P(n) ⇒ P(n+1)

But for what values of n? Need to prove it for all n≥1 since that is base case. Look at the line:

**Color (H1) = color (H2 …. Hn) = color (Hn+1)**

**Q**. what happens when n=1?

**A.** H2 …. Hn is empty! Argument is bogus. There is no set of horses in the middle. So the argument **only works for n ≥ 2, not n=1!**

Proof that P(n) ⇒ P(n+1) only holds for n ≥ 2

**So we proved:**

**P(1) , P(2) ⇒ P(3), P(3) ⇒ P(4), ……**

**Missing link**

Note that we proved that P(2) ⇒ P(3) but that does not mean P(3) is true since we never showed P(1) ⇒ P(2).

In fact P(2) is false as are P(3), P(4), and so on.

Valuable lessons: When doing inductive step, make sure that it holds for all n ≥ base case. When using … notation, make sure to keep track of when set is empty. “…” notation seems so reasonable and so easy to use and so clear, but also easily leads to errors.

Questions?

**Quit here if no time**

**Explain starting induction at n = 2.**

**Base Case Fails.**

Questions?